A charged rotating cylindrical shell[1]

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We give an example of a spacetime having an infinite thin rotating cylindrical shell constituted by a charged perfect fluid as a source. As the interior of the shell the Bonnor–Melvin universe is considered, while its exterior is represented by Datta–Raychaudhuri spacetime. We discuss the energy conditions and we show that our spacetime contains closed timelike curves. Trajectories of charged test particles both inside and outside the cylinder are also examined. Expression for the angular velocity of a circular motion inside the cylinder is given.

KEY WORDS: matching of solutions, perfect fluid, electromagnetic field, test particles

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1 Introduction

Cylindrically symmetric sources of gravitational field have been intensively studied during the whole development of general relativity. Although they are unbounded and can not represent real objects creating gravitational field, they constitute framework for investigation of spacetimes with a high degree of symmetry in the field of exact as well as numerical solutions to the Einstein equations [1, 2, 3]. Cylindrically symmetric sources have become of a great significance in context of relativistic cosmology [2, 4] and they provide an important tool for examining dynamical models, e.g. in cases with presence of gravitational waves [5]. They have been connected with considerations taking quantum gravity and probe trajectories in context of string theory into account as well [6].

A number of papers have been concerned with a rotating cylinder in general relativity ([2, 7] and references therein). In particular, both rotating and nonrotating cylindrical thin shells have been studied [8, 9]. In the works [10] various kinds of shell sources for static Levi–Civita and Lewis spacetime respectively have been discussed. Charged generalizations of the Levi–Civita spacetime and their shell sources have been studied in [11].

The outline of the paper is the following. Section 2 characterizes the physical set up – matching the Bonnor–Melvin magnetic universe [12] to solution of Datta and Raychaudhuri [13]. In section 3, by using of the Israel–Kuchař junction conditions [14, 15], we put forward a spacetime with an infinite rotating cylindrical shell, built up from a charged perfect fluid acting as a source of the gravitational and electromagnetic field. Afterward, in section 4, we discuss some attributes of the solution, in particular a question of the chronology violation in the spacetime and the energy conditions. Finally in section 5 the question is addressed what is a qualitative behavior of trajectories of test particles, either charged or uncharged.

2 An interior and an exterior spacetime and a shell

The goal of the paper is to obtain a spacetime having a cylindrical shell as its source. The spacetime arises by matching a conveniently chosen interior spacetime to a suitably chosen exterior one. From reasons given bellow the interior portion is to be constituted by (a portion of) the Bonnor–Melvin magnetic universe [12, 1] while as the exterior one we have chosen solution of Datta and Raychaudhuri [13]. It is worth to note that these two spacetimes share the same symmetries, namely they are cylindrically symmetric and stationary.

The Bonnor–Melvin magnetic universe (BM) describes a static electrovacuum spacetime and we restrict ourselves to values of radial coordinate $r \in (0, r_B]$. The metric and the electromagnetic potential of this spacetime have

the form[1]

$$\mathrm{d}s^2 = -(1-K^2r^2)^{-2}\mathrm{d}t^2 + (1-K^2r^2)r^2\mathrm{d}\phi^2 + (1-K^2r^2)^{-2}\mathrm{d}z^2 + (1-K^2r^2)^{-5}\mathrm{d}r^2,$$
$$A = Kr^2\mathrm{d}\phi \tag{2.1}$$

with K being a constant, $t \in (-\infty, \infty)$, $z \in (-\infty, \infty)$ and ϕ being 2π-periodic angular coordinate. Correctness of the signature requires $r_B < \frac{1}{|K|}$. The choice of BM spacetime is motivated by the following facts. If we eliminate the gravitational coupling of an electromagnetic field in this spacetime by setting the gravitational constant equal to zero, we get the Minkowski spacetime with a homogeneous magnetic field of the magnitude $2K$ pointing in z direction. But this is exactly the interior of a rotating charged cylindrical shell in special relativity theory. Furthermore, BM spacetime has no singularity on the rotation axis. In addition, it satisfies the elementary flatness condition.

The exterior of the cylinder is constituted by Datta and Raychaudhuri (DR) solution for values of radial coordinate $r \in [r_D, \infty)$. The metric and the electromagnetic potential of DR spacetime have the form

$$\begin{aligned}\mathrm{d}s^2 = {} & (-U^2(4r^2 + \lambda r\ln(r)) + 2U\Omega r)\mathrm{d}t^2 + (-H^2(4r^2 + \lambda r\ln(r)) + 2HDr)\mathrm{d}\phi^2 + \\ & + 2(-UH(4r^2 + \lambda r\ln(r)) + (\Omega H + UD)r)\mathrm{d}t\mathrm{d}\phi \\ & + V^2 r^{-\frac{1}{2}}\mathrm{d}z^2 + r^{-\frac{1}{2}}\mathrm{d}r^2 \ , \end{aligned} \tag{2.2}$$
$$A = -r(U\mathrm{d}t + H\mathrm{d}\phi) \ , \tag{2.3}$$

where $U, H, \Omega, D, V, \lambda$ are constants. The coordinates t, z have the same range as in BM and ϕ is assumed to be 2π–periodic angular coordinate again.

DR is a non–static spacetime with null electromagnetic field ($F_{\mu\nu}F^{\mu\nu} = 0$), which contains a radial electric field and a magnetic field pointing in z direction. DR solution is a charged generalization of Van Stockum solution (see bellow). The metric and the electromagnetic potential (2.2) and (2.3) can be locally obtained from the following metric and electromagnetic potential

$$\mathrm{d}s^2 = -(4r^2 + \lambda r\ln r)\mathrm{d}\bar{t}^2 + 2r\mathrm{d}\bar{t}\mathrm{d}\bar{\phi} + r^{-\frac{1}{2}}\mathrm{d}\bar{z}^2 + r^{-\frac{1}{2}}\mathrm{d}r^2 \ , \tag{2.4}$$
$$A = -r\mathrm{d}\bar{t} \tag{2.5}$$

by the linear transformation

$$\begin{pmatrix} t \\ \phi \\ z \end{pmatrix} = \begin{pmatrix} U & H & 0 \\ \Omega & D & 0 \\ 0 & 0 & V \end{pmatrix}^{-1} \begin{pmatrix} \bar{t} \\ \bar{\phi} \\ \bar{z} \end{pmatrix}. \tag{2.6}$$

The transformation matrix (2.6) have to be regular. If the coordinates with bars are used in spacetime (2.2), (2.3), then $\bar{\phi}$ isn't angular generally.

The boundary between BM and DR spacetimes is given by the equation $r = r_B$ in BM and by the equation $r = r_D$ in DR. These hypersurfaces are

[1]In the paper the natural units has been used, i.e. $c = G = 1$ and $\mu_0 = 4\pi$.

joined by identification of the points with the same coordinates t, ϕ, z. Let us call $T,\ \Phi,\ Z$ the coordinates that arise from the coordinate functions $t,\ \phi,\ z$ by restriction their domain to the boundary.

Let a charged rotating cylindrical shell built up from a perfect fluid be situated at the boundary. The shell surface energy – momentum tensor has the form ([15]) [2]

$$t_{ab} = (p+\rho)u_a u_b + p g_{ab}, \tag{2.7}$$

with ρ being rest surface mass density, p rest surface pressure on the shell and g_{ab} being the components of the induced metric at the boundary. The values of $\rho,\ p$ do not depend on a position at the shell. Particles of the shell move with the 4–velocity

$$\mathbf{u} = \frac{\frac{\partial_T}{\sqrt{-g_{TT}}} + v\frac{\partial_\Phi}{\sqrt{g_{\Phi\Phi}}}}{\sqrt{1-v^2}}\ , \tag{2.8}$$

where v is the velocity of the shell particles in ϕ direction as measured by observers moving on curves $\{\Phi,\ Z\} = \text{const}$. The condition $|v| < 1$ must be satisfied in order that $\mathbf{u}$ be timelike.

3 Junction conditions and their solution

We denote g^+ and g^- the induced metric at the boundary from side of DR spacetime and BM spacetime respectively. Similarly, let us denote k^+ and k^- the external curvature of the boundary with respect to outward normal vector field. Israel junction conditions for gravitational field [14] then become[3]

$$g^+_{ab} = g^-_{ab}\ , \qquad k^+_{ab} - k^-_{ab} = 8\pi(t_{ab} - \frac{1}{2}t g_{ab}), \tag{3.9}$$

where $g_{ab} := g^+_{ab} = g^-_{ab}$. Furthermore let σ be the rest surface charge density which does not depend on a position at the shell, $\mathbf{s} = \sigma\mathbf{u}$ be the surface current density of the shell, $\mathbf{n}$ outward normal to the boundary and F^+ and F^- the electromagnetic field tensor in DR and BM spacetime. Kuchař junction conditions for the electromagnetic field [15] have the form

$$F^+(\partial_a, \mathbf{n}) - F^-(\partial_a, \mathbf{n}) = 4\pi s_a, \quad F^+(\partial_a, \partial_b) = F^-(\partial_a, \partial_b). \tag{3.10}$$

The conditions (3.9), (3.10) lead to a system of ten algebraic equations for thirteen unknowns in our case. By solving them one can express quantities $r_D, U, \Omega, H, D, V, \lambda, p, \rho, \sigma$ in terms of functions of K, r_B, v, h, where $h = \pm 1$ arises from solving of a certain quadratic equation. These functions take the form

$$r_D^{-\frac{3}{4}} = \frac{4}{3}\frac{1-4K^2 r_B^2}{r_B X^{-\frac{3}{2}}(1-hv)}\ , \tag{3.11a}$$

[2] Possible values of indexes a, b are $T,\ \Phi,\ Z$.

[3] In the paper the sign conventions of [16] have been used.

$$hU = \frac{2r_D^{-\frac{1}{4}}KX}{1-hv} \ , \tag{3.11b}$$

$$H = -\frac{2r_B r_D^{-\frac{1}{4}}KX^{\frac{5}{2}}}{1-hv} \ , \tag{3.11c}$$

$$h\Omega = -\frac{1-4K^2r_B^2}{3Kr_BX^{\frac{3}{2}}} + \frac{hv(1-4K^2r_B^2)}{2Kr_BX^{\frac{3}{2}}(1-hv)}\ln r_D - \frac{3Kr_BX^{-\frac{1}{2}}}{1-4K^2r_B^2}(\ln r_D - 1) \ , \tag{3.11d}$$

$$D = -\frac{2}{3K}(1-4K^2r_B^2) - r_BX^{\frac{3}{2}}h\Omega \ , \tag{3.11e}$$

$$V = r_D^{\frac{1}{4}}X^{-1} \ , \tag{3.11f}$$

$$\lambda = \frac{1}{hU}\left(\frac{hv(1-4K^2r_B^2)}{Kr_BX^{\frac{3}{2}}(1-hv)} - \frac{6Kr_BX^{-\frac{1}{2}}}{1-4K^2r_B^2}\right) \ , \tag{3.11g}$$

$$16\pi\rho = r_D^{-\frac{3}{4}} + 8K^2r_BX^{\frac{3}{2}} \ , \tag{3.11h}$$

$$16\pi(p+\rho) = \frac{2X^{\frac{3}{2}}(1-4K^2r_B^2)(1+hv)}{r_B(1-hv)} \ , \tag{3.11i}$$

$$4\pi h\sigma = -\frac{2KX^2}{1-hv}(1-v^2)^{\frac{1}{2}} \ , \tag{3.11j}$$

where $X := 1 - K^2r_B^2$. The condition for correct signature of (2.1) guarantees that $X > 0$. Moreover, the inequalities

$$\mid Kr_B \mid < \frac{1}{2} \ , \qquad K \neq 0 \tag{3.12}$$

have to be satisfied, to get positive right side of (3.11a) and to prevent infinities. The regularity of (2.6) requires $(UD - H\Omega)V = \frac{4}{3}\,\frac{1-4K^2r_B^2}{v-h} \neq 0$. This is satisfied, because (3.12) holds. Note that the line element (2.2) remains unaffected by a transformation $v \to -v,\ \phi \to -\phi,\ h \to -h$ instead of expected $v \to -v,\ \phi \to -\phi$. But we can redefine $\bar{h} = h\ \mathrm{sign}(v)$ for $v \neq 0$. Then $hv = \bar{h}|v|$ holds and keeping $\bar{h}$ unchanged we get the expected symmetry. A connection between $\bar{h}$ and physical properties of the obtained spacetime can be seen for example from a relation $\bar{h} = -\mathrm{sign}(s_\Phi/B_Z)$ (for $s_\Phi \neq 0$), where s_Φ is Φ–component of the surface current density and B_Z is Z–component of the magnetic field at the shell taken from an arbitrary side of the shell. Thus a change of $\bar{h}$ implies a change of a direction in which the magnetic field points out near the shell with respect to the current on the shell.

The uncharged spacetime can be obtained from the charged one by taking a limit $K \to 0$. Inside the shell it leads to the Minkowski spacetime. The same limit performed outside the shell yields

$$\mathrm{d}s^2 = -\mathrm{sign}(hv)r\ln(r)\mathrm{d}\tilde{t}^2 + 2r\mathrm{d}\tilde{\phi}\mathrm{d}\tilde{t} + r^{-\frac{1}{2}}\left(\mathrm{d}\tilde{z}^2 + \mathrm{d}r^2\right) \ ,$$

which coincides with the Van Stockum metric [1].

4 Physical properties of the matched spacetime

4.1 Energy conditions

As one can see from (3.11h) and (3.11i), the weak energy condition, $\rho \geq 0, \rho + p \geq 0$, is satisfied for all allowed values of the free parameters.

The strong energy condition, $p + \rho \geq 0$, $\rho + 2p \geq 0$, gives the following restriction of the free parameters

$$K^2 r_B^2 \leq \frac{1}{2}\ \frac{2 + 3hv}{7 + 3hv}\ . \tag{4.13}$$

The dominant energy condition is equivalent to the weak energy condition plus the inequality $\rho - p \geq 0$ which reads

$$K^2 r_B^2 \geq \frac{1}{4}\ \frac{3hv - 1}{5 - 3hv}\ . \tag{4.14}$$

4.2 Linear charge density

The linear charge density of the shell as measured by an observer moving with 4–velocity $\mathbf{u}_0 = \partial_T / \sqrt{-g_{TT}}$ becomes equal to

$$q = 2\pi \sqrt{g_{\Phi\Phi}} (-\mathbf{s}.\mathbf{u}_0) = \frac{hKr_B X^{\frac{5}{2}}}{hv - 1}\ . \tag{4.15}$$

If we are given v, r_B, h and want to find a dependence of some quantity R on q, we may express this dependence parametrically as $R = R(K), q = q(K)$. But a change in K with v, r_B, h kept constant implies also a change of the mass parameters ρ, p and a change of the circumference of the shell $2\pi\sqrt{g_{\Phi\Phi}}$. Consequently one is not able to decide whether the changes of R have a physical origin in changes of the charge or the mass. However, the following equations hold

$$\frac{\partial p}{\partial K}\ |_{K=0} = 0\ , \quad \frac{\partial \rho}{\partial K}\ |_{K=0} = 0\ , \quad \frac{\partial}{\partial K}\sqrt{g_{\Phi\Phi}}\ |_{K=0} = 0\ , \quad \frac{\partial q}{\partial K}\ |_{K=0} \neq 0\ . \tag{4.16}$$

Formulae (4.16) show that for small K (and consequently for small q) the mass quantities expressed in terms of functions of q do not change significantly. The dependence $m(q)$, where $m = 2\pi\sqrt{g_{\Phi\Phi}}\ t(\mathbf{u}_0, \mathbf{u}_0)$ is the linear mass density measured by an observer with the velocity $\mathbf{u}_0$, is illustrated in figure 1.

4.3 Closed timelike curves

In our spacetime the sign of the metric coefficient $g_{\phi\phi}(r)$ is crucial for existence or nonexistence of closed timelike curves (CTC). Our spacetime does not contain any horizons. This statement is essential for further considerations and it will be proved in section 5. Because there are no horizons (generated by timelike geodesics), if there exists a point p with the property that $g_{\phi\phi}(p) < 0$, then for

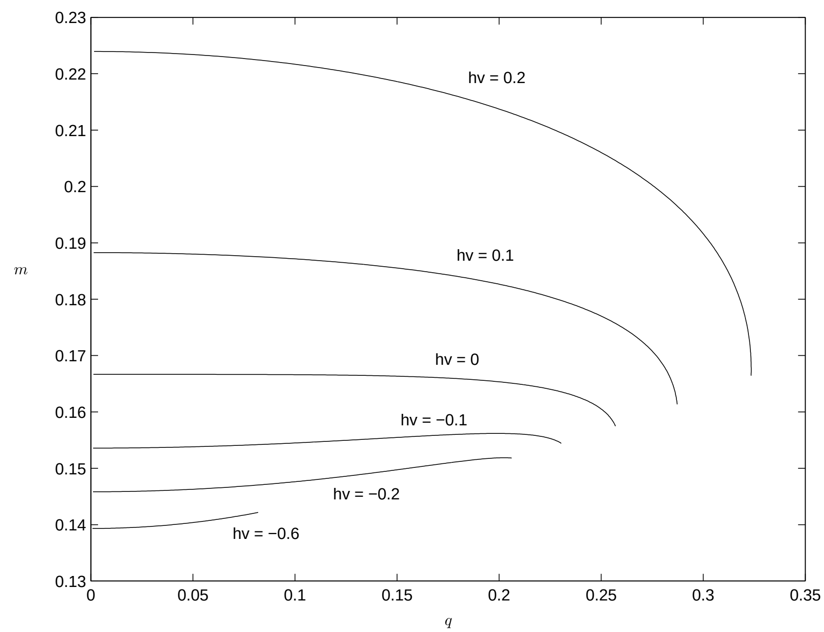

Figure 1: The dependence $m(q)$ for various hv. The mass m is for negative q obtained by $m(q) = m(-q)$. The parameter hKr_B satisfies the energy conditions.

each point q of the spacetime there exists CTC which goes through q. However, each such curve must intersect the region where $g_{\phi\phi}(r) < 0$. If $g_{\phi\phi}(r) > 0$ in the entire spacetime, there are no CTCs.

It holds $g_{\phi\phi}(r) = (1-K^2r^2)r^2 > 0$ inside the shell, since it is always $r < \frac{1}{|K|}$ in BM spacetime. Outside the shell we have $g_{\phi\phi}(r) = -H^2(4r^2 + \lambda r\ln(r)) + 2HDr$. To determine the sign of $g_{\phi\phi}$, it is sufficient to examine the function $\frac{g_{\phi\phi}(r)}{r}$. One obtains

$$\lim_{r\to\infty} \frac{g_{\phi\phi}}{r} = -\infty \ , \ \frac{g_{\phi\phi}}{r}(r_D) > 0 \ , \ \left(\frac{g_{\phi\phi}}{r}\right)' = -H^2(4 + \frac{\lambda}{r}) \ , \ \left(\frac{g_{\phi\phi}}{r}\right)'' = H^2\frac{\lambda}{r^2} \ ,$$

from which one can infer that there always exists $r_C \in (r_D, \infty)$ such that $g_{\phi\phi} > 0$ for $r \in (r_D, r_C)$, $g_{\phi\phi}(r_C) = 0$ and $g_{\phi\phi} < 0$ for $r \in (r_C, \infty)$. Consequently, for each point q of our spacetime and for all allowed values of the free parameters there exists CTC which passes through q. Nevertheless, all CTCs must intersect the region where $r > r_C$. If $hv > 0$, then the same conclusion is true for uncharged limit of spacetime, obtained by taking the limit $K \to 0$. On the other hand, if $hv \leq 0$ and $K \to 0$, then $g_{\phi\phi}(r)$ is positive in entire spacetime, so the causality violation is avoided in this case.

The dependence of the proper radial distance $R_C = \int_{r_D}^{r_C} \sqrt{g_{rr}}dr$ on the linear charge density q is depicted in figure 2 in case $hv \leq 0, r_B = 1$. The case

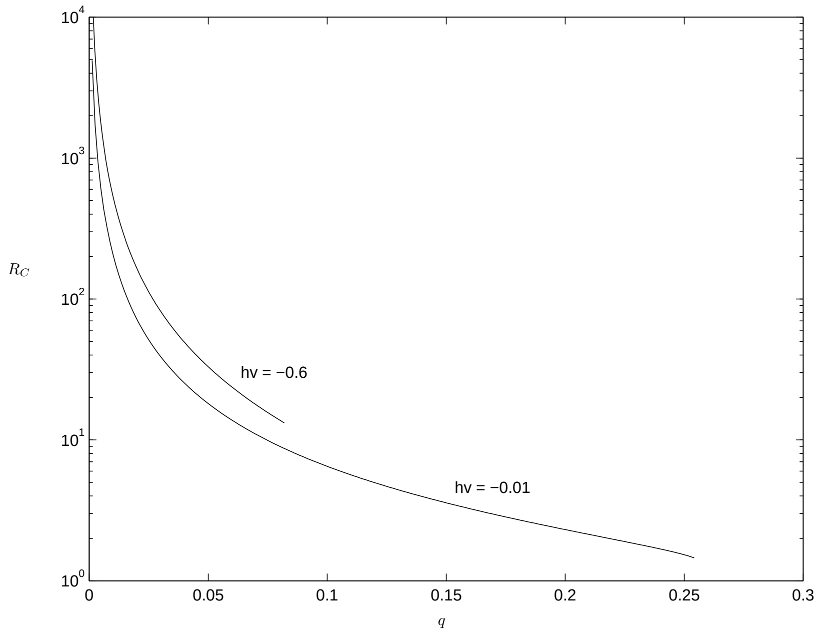

Figure 2: Proper radial distance between r_D and r_C for $r_B = 1$ and various $hv \leq 0$. R_C blows up when $q \to 0$. It corresponds to the fact that the uncharged limit contains no CTCs.

$hv > 0, r_B = 1$ is shown in figure 3. The quantities R_C, q as functions of the free parameters are of the form $R_C((hK)^2, r_B, hv)$ and $q(hKr_B, hv)$. The curves in figure 2 and figure 3 plotted for given r_B and hv are given parametrically. The parameter hK runs through negative values satisfying the conditions (4.13) and (4.14). For given hK the value of q is computed from (4.15), the value of r_C is found by numeric solution of the equation $g_{\phi\phi}(r) = 0$. Since R_C is even function of hK and q is odd, positive values of hK give curves which differ from the ones in figure 2 and figure 3 only by the substitution $[q, R_C] \to [-q, R_C]$. Since $R_C((hK/x)^2, xr_B, hv) = xR_C((hK)^2, r_B, hv)$ holds for $x > 0$, the curves for $r_B \neq 1$ can be obtained from the corresponding curves in figure 2 and figure 3 by $[q, R_C] \to [q, r_B R_C]$.

One can see that for small values of $|q|$, R_C decreases with increasing values of $|q|$, which is in a qualitative agreement with [4].

4.4 Electromagnetic field

By normalizing vectors of the coordinate base inside the cylinder one gets an orthonormal basis. The only non–zero independent component of the interior electromagnetic field tensor F in this base is $F(\frac{\partial_r}{\sqrt{g_{rr}}}, \frac{\partial_\phi}{\sqrt{g_{\phi\phi}}}) = B_{\hat{z}} = 2K(1 - K^2r^2)^2$.

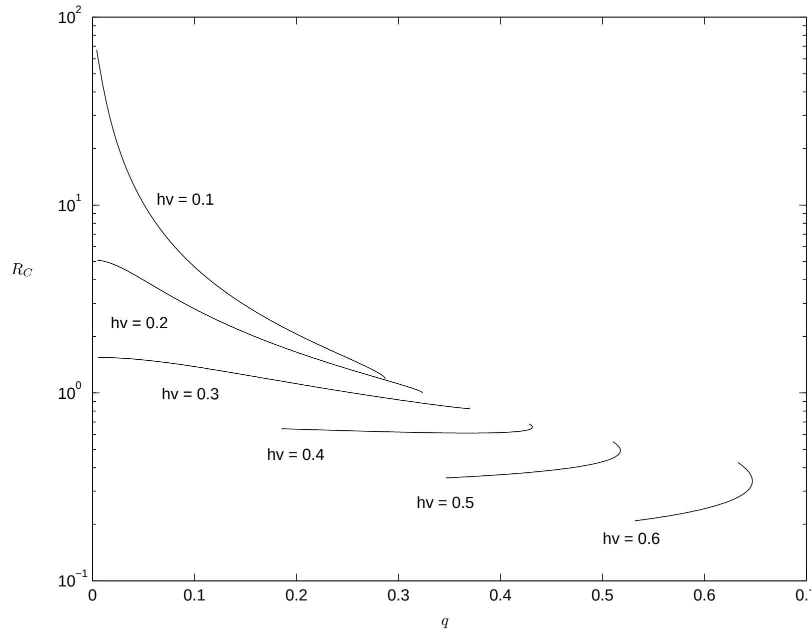

Figure 3: Proper radial distance between r_D and r_C for $r_B = 1$ and various $hv > 0$. R_C has a finite limit when $q \to 0$. It corresponds to the fact that the uncharged limit contains CTCs too.

Outside the cylinder we can construct the orthonormal basis

$$\mathbf{v} = \frac{\sqrt{|g_{\phi\phi}|}(\partial_t - \frac{g_{t\phi}}{g_{\phi\phi}}\partial_\phi)}{\sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}} \ , \quad \mathbf{w} = \frac{\partial_\phi}{\sqrt{|g_{\phi\phi}|}} \ , \quad \mathbf{z} = \frac{\partial_z}{\sqrt{g_{zz}}} \ , \quad \mathbf{n} = \frac{\partial_r}{\sqrt{g_{rr}}} \ .$$

Note that $g_{t\phi}^2 - g_{tt}g_{\phi\phi} = (UD - H\Omega)^2 r^2$ is always positive. The vector field $\mathbf{v}$ is timelike for $g_{\phi\phi} > 0$ while $\mathbf{w}$ is spacelike [4]. If $g_{\phi\phi} < 0$ the fields $\mathbf{v}$ and $\mathbf{w}$ will change their roles. There are two non–zero independent components of the exterior electromagnetic tensor

$$F(\mathbf{n},\mathbf{v}) = hHr^{\frac{1}{4}}\frac{\mathrm{sign}(g_{\phi\phi})}{\sqrt{|g_{\phi\phi}|}} \ , \quad F(\mathbf{n},\mathbf{w}) = -\frac{Hr^{\frac{1}{4}}}{\sqrt{|g_{\phi\phi}|}} \ .$$

For $g_{\phi\phi} > 0$ one has $E_{\hat{r}} = F(\mathbf{n},\mathbf{v})$ and $B_{\hat{z}} = F(\mathbf{n},\mathbf{w})$ and for $g_{\phi\phi} < 0$ we have $E_{\hat{r}} = F(\mathbf{n},\mathbf{w})$ and $B_{\hat{z}} = F(\mathbf{n},\mathbf{v})$. Both components vanish in infinity and their magnitudes diverge as r approaches r_C. In the special relativity theory in order that the electromagnetic field be vanishing in infinity, one has to put the magnetic field outside the cylinder equal to zero, because of its homogeneity, but one can see that it is not the case in the general relativity.

[4]Observer with 4–velocity $\mathbf{v}$ is called "locally nonrotating".

5 Geodesic completeness and motion of test particles

In this section we aim to study briefly a motion of test particles that are generally charged, carrying a charge e. The trajectory is a solution of the equation

$$\nabla_{\mathbf{u}} u^\alpha = -\nu F_\beta{}^\alpha u^\beta \ ,$$

where constant ν stands for $\frac{e}{m}$ in case of a massive charged particle, while ϵ equals 1 or 0 depending on whether we examine timelike or null curves (geodesics in the latter case). The velocity vector field $\mathbf{u} = \frac{\mathrm{d}}{\mathrm{d}s} = u^\alpha \partial_\alpha$ is assumed to be normalized, $\mathbf{u} \cdot \mathbf{u} = -\epsilon$. We divide the discussion into two classes.

5.1 Test particles in BM spacetime

Because of the high degree of symmetry we have three conserved quantities – an energy E, an angular momentum L and a momentum along z–axis P_z. These integration constants arise after first integration of the equations of motion for a test particle of a mass m, carrying a charge e. We find

$$\frac{\mathrm{d}t}{\mathrm{d}s} = E(1 - K^2 r^2)^2 \ , \tag{5.17a}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}s} = \frac{L - K\nu r^2}{r^2(1 - K^2 r^2)} \ , \tag{5.17b}$$

$$\frac{\mathrm{d}z}{\mathrm{d}s} = P_z (1 - K^2 r^2)^2 \ . \tag{5.17c}$$

Radial coordinate fulfills the following equation

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 = V_{\mathrm{eff}} \ , \tag{5.18}$$

with the effective potential V_{eff} given by

$$V_{\mathrm{eff}} = \frac{1}{2r^2}(1 - K^2 r^2)^4 \Big[\left(E^2 - P_z^2\right) r^2 \left(1 - K^2 r^2\right)^3 - \\ - \epsilon r^2 \left(1 - K^2 r^2\right) - \left(L - K\nu r^2\right)^2 \Big] \ . \tag{5.19}$$

The motion of a test particle is clearly restricted to values of r for which (5.19) (and consequently the expression in square brackets of (5.19)) is non–negative. First of all this means that $F^2 \equiv E^2 - P_z^2 \geq 0$. By using the estimate $x(1 - x) \leq \frac{1}{4}$ for $0 \leq x \leq 1$, where $x \equiv K^2 r^2$, one can make a rough conclusion that a motion of a test particle is forbidden at least for the following ranges[5] of the integration constants ($G \equiv KL$):

i for $F^2 \leq 4(G - \nu)^2 + \epsilon$ if $G \geq \nu \geq 0$ or $0 \geq \nu \geq G$.

[5] This is an estimate, the exact ranges are wider.

ii for $F^2 \leq 4G^2 + \epsilon$ if $G \geq 0 \geq \nu$ or $\nu \geq 0 \geq G$.

iii for $F^2 \leq \epsilon$ if $\nu \geq G \geq 0$ or $0 \geq G \geq \nu$.

Although it is not generally possible to integrate (5.19) in terms of elementary functions, one can obtain an exact expression for the angular velocity $\omega \equiv \frac{\mathrm{d}\phi}{\mathrm{d}\tau}$ of a test particle moving on a circular trajectory $z = \mathrm{const}$, $r = r_0$. In units SI it reads

$$\omega = \frac{K}{1 - 4\mathcal{G}K^2 r_0^2} \left(-\nu \pm \sqrt{\nu^2 + 2\mathcal{G}c^2 \frac{1 - 4\mathcal{G}K^2 r_0^2}{1 - \mathcal{G}K^2 r_0^2}} \right) , \tag{5.20}$$

where $\mathcal{G} = \frac{4\pi G}{\mu_0 c^4} \approx 8,15.10^{-38}\, T^{-2} m^{-2}$. If the gravity is switched off ($G = 0$) one obtains the classical cyclotron frequency $\omega_{STR} = -2K\nu = -\frac{B_z e}{m}$, since it is $B_z = 2K$ for $G = 0$. Expanding (5.20) in $\mathcal{G}$ we get the approximate expression for the relative deviation of cyclotron frequency caused by gravity

$$\frac{\omega - \omega_{STR}}{\omega_{STR}} \approx \mathcal{G}\left(\frac{c^2}{2\nu^2} + 4K^2 r_0^2\right) .$$

To see that the gravity has only a small influence on cyclotron frequency in usual conditions, let us evaluate how much ω differs from ω_{STR}. For an electron, $K = 10T$ and $r_0 = 0,1m$, one has the relative deviation of order 10^{-37}. Some trajectories of the charged test particles are depicted in figure 4.

5.2 Test particles in DR spacetime

For the sake of technical simplicity, we will consider the line element (2.4). Just like in BM spacetime we are led to the following equations of motion (with bars over coordinates omitted)

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{L}{r} , \tag{5.21a}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}s} = \nu - \frac{E}{r} + L\left(4 + \lambda \frac{\ln r}{r}\right) , \tag{5.21b}$$

$$\frac{\mathrm{d}z}{\mathrm{d}s} = \sqrt{r} P_z , \tag{5.21c}$$

along with the following effective potential governing the radial motion of a test particle

$$V_{\mathrm{eff}} = -\frac{1}{2}\sqrt{r}\left[4L^2 + \epsilon + 2\nu L + \sqrt{r}P_z^2 - \frac{2E}{r}L + L^2\lambda\frac{\ln r}{r}\right] . \tag{5.22}$$

Since an integration of (5.22) is not possible explicitly in general, we will discuss here a basic behavior only. First of all, irrespective of values of the integration constants, V_{eff} tends to minus infinity as r goes to infinity. This means that the test particles radial motion is bounded and it is forbidden behind a certain critical radius depending on a particular path.

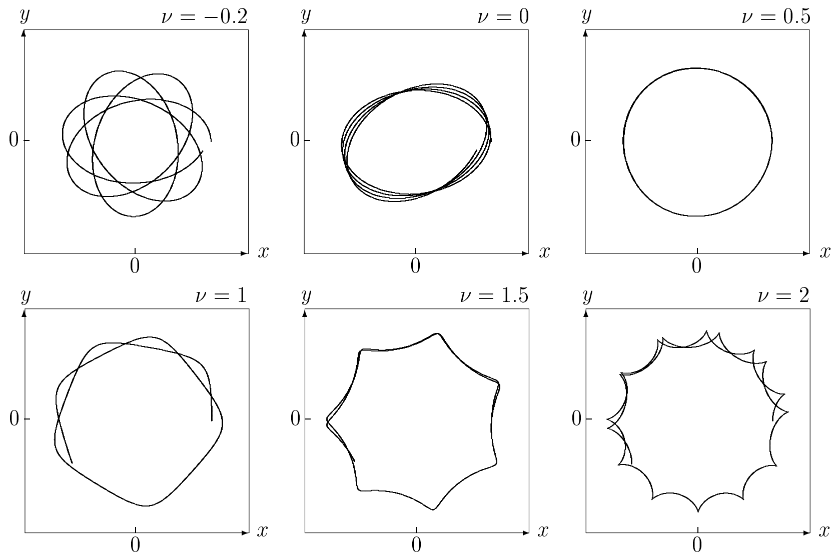

Figure 4: Some trajectories of charged test particles inside the cylinder with z = const. The coordinates x, y depend on r, ϕ as usually: $x = r\cos\phi,\ y = r\sin\phi$. The initial values and the value of K are same for all the pictures. In the natural units we have $\frac{\mathrm{d}r}{d\tau}(0) = \frac{\mathrm{d}z}{d\tau}(0) = 0,\ \frac{\mathrm{d}\phi}{d\tau}(0) = 1,\ r_0 = 0.1,\ K = 1$. The values of ν are written in the figure.

On the other hand, if $\lambda > 0$ there is r_1 such that $V_{\text{eff}} > 0$ for $r < r_1$. If $\lambda < 0$ the situation gets more complicated depending on values of others integration constants. In this case both possibilities can occur. Either a range $r \in [r_0, r_1]$ exists in which a test particle motion is permitted, or it may become that the motion will be forbidden entirely. We shall not be concerned with an investigation under which circumstances either behavior could occur further. Instead, provided the test particles can move and the inequality $r_1 > r_D$ is satisfied, we will show that the spacetime is complete.

Because according to our assumption r is bounded both from bellow and above, it follows from the equations (5.21a)-(5.21c) that the terms $\frac{\mathrm{d}t}{\mathrm{d}s}$, $\frac{\mathrm{d}\phi}{\mathrm{d}s}$ and $\frac{\mathrm{d}z}{\mathrm{d}s}$ can be bounded by suitable chosen constants A, B and C as $\left|\frac{\mathrm{d}z}{\mathrm{d}s}\right| < A$ and so on for t and ϕ. Thus the test particle can escape to infinity in z or t direction only in infinite s showing that every geodesic[6] in our matched spacetime can be continued to an arbitrary value.

Note that the component g_{00} of the metric tensor (2.4) vanishes for $r = 0$. Moreover, if λ is positive one has one additional root, and if $\lambda < -4\mathrm{e}$ (e is Euler constant), we have two additional roots of g_{00}. As we shall see immediately, $r = 0$ represents the physical singularity while the two additional roots do not.

[6]The geodesics are obtained by setting e equal to zero.

Since in stationary spacetimes frequencies of a light signal measured at two distinct points p and q of the geodesic along which the signal moves are related by $\frac{\omega(p)}{\omega(q)} = \sqrt{\frac{g_{00}(q)}{g_{00}(p)}}$, one finds the physical interpretation of these two roots: they corresponds to hypersurfaces of infinite red shift. Being timelike, these hypersurfaces are not horizons.

A direct computation of the Riemann curvature tensor invariants shows that the rotation axis $r = 0$ is the only intrinsic singularity. For instance the first non–trivial[7] curvature invariant is equal to

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{3}{4r^3} \ .$$

Generally, dangerous terms involved in the components of the Riemann tensor, that could be possibly responsible for singularities, are proportional to r^{k_1}, $r^{k_2}\ln r$, where the constants k_1 and k_2 acquire negative values in all but one case, namely when $k_1 = 1/2$. It can be seen that their combination resulting from a computation of any Riemann curvature invariant is tame for r going to infinity. Thus the rotation axis constitutes the only physical singularity, but because we have truncated DR spacetime on r_D and restricted ourselves to region $r \in [r_D, \infty)$, we are left with entirely non–singular spacetime.

6 Conclusion

We found spacetime, where infinite rotating cylindrical shell from charged perfect fluid acts as a source. Bonnor–Melvin magnetic universe has been used as the interior part of the cylinder and Datta and Raychaudhuri spacetime as the exterior one. Because of the junction conditions, the metric, the electromagnetic potential and the shell parameters have been expressed as functions of four free parameters K, r_B, v, h, where K is closely connected with the magnetic field inside the shell, r_B is the value of the radial coordinate at witch the interior spacetime has been cut off (i.e. where the shell is located), v is Φ–component of the shell particles velocity and h can have only values ± 1. The question was examined in what ranges of the parameters the energy conditions are satisfied.

The spacetime found contains closed timelike curves for all allowed values of the free parameters. Also it was shown that the radial distance between the shell and the radius r_C behind which each CTC must pass, decreases with an increasing absolute value of the linear charge density $|q|$ at the shell for small values of $|q|$. Finally an investigation of the test particles trajectories, either charged or uncharged, was carried out both for the interior as well as exterior region. In particular it was found the the trajectories are always radially bounded and that the resulting spacetime is free of physical singularities.

[7]The simplest invariant, Ricci curvature, clearly vanishes identically and it also holds for $R_{\mu\nu}R^{\mu\nu}$.

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